Noether's Conservation Laws - smooth and discrete

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Relating to joint work with Tania Gonçalves, Tristan Pryer, Ana Rojo-Echeburúa, Peter Hydon, Michele Zadra, Linyu Peng

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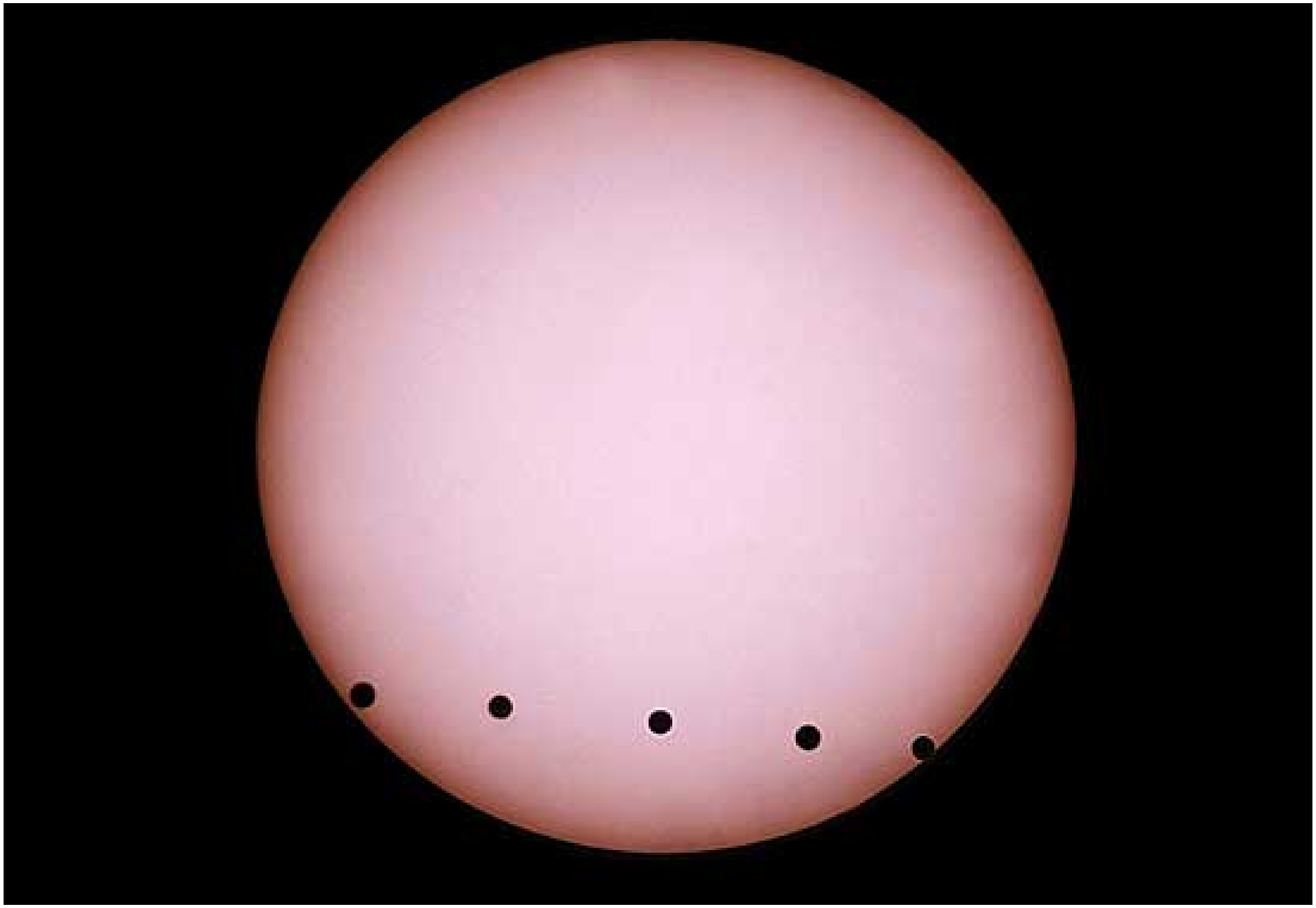


A time lapse photo of the transit of venus across the sun

The transit of venus across the sun at sunset

Satellite image of a cyclone

This talk:

Smooth Case – use invariants and moving frames.

- Expose the structure of equations and laws.
- Combat expression swell.

Discrete Case – embedding the physics via the Lie symmetry into the numerics.

- Finite Difference – structure mirrors that of the smooth case.
- Finite Element – very different look and feel.

Running Example: projective $SL(2)$ action

$$g \cdot x = x, \qquad g \cdot u = \frac{au + b}{cu + d}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

Via the chain rule, induce an action on u_x etc:

$$g \cdot u_x = \frac{\partial(g \cdot u)}{\partial(g \cdot x)} = \frac{u_x}{(cu + d)^2}$$

Lowest order invariant is the so-called Schwarzian derivative,

$$V = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} := \{u; x\}.$$

Suppose our Lagrangian is

$$L(x,u,u_x,\ldots,u_{xxxxx})\,\mathrm{d}x = \left(\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\{u;x\}\right)^2 + \frac{1}{2}\{u;x\}^2\right)\,\mathrm{d}x.$$

Then this is invariant under the induced action of $SL(2)$ and there are three first integrals, one for each dimension of $SL(2)$.

The Euler–Lagrange equation has order 10, and one of the first integrals is:

Using Maple's `DifferentialGeometry` package

> $Noether\left(\frac{1}{2}\cdot u[\,]^2\cdot D\_u[\,],\ lambda2\right);$

$$
\begin{aligned}
&-\frac{1}{4}\,\frac{1}{u_1^9}\Big(2\,u_{[\,]}^2\,u_1^7\,u_{1,1,1,1,1}+4\,u_{[\,]}^2\,u_1^7\,u_{1,1,1,1,1,1,1,1,1}-8\,u_{[\,]}^2\,u_1^6\,u_{1,1}\,u_{1,1,1,1}-32\,u_{[\,]}^2\,u_1^6\,u_{1,1}\,u_{1,1,1,1,1,1,1,1}-6 \qquad \text{(1}\\
&\quad u_{[\,]}^2\,u_1^6\,u_{1,1,1}^2-132\,u_{[\,]}^2\,u_1^6\,u_{1,1,1}\,u_{1,1,1,1,1,1,1}-284\,u_{[\,]}^2\,u_1^6\,u_{1,1,1,1}\,u_{1,1,1,1,1,1}-180\,u_{[\,]}^2\,u_1^6\,u_{1,1,1,1,1}^2+21\,u_{[\,]}^2\\
&\quad u_1^5\,u_{1,1}^2\,u_{1,1,1}+240\,u_{[\,]}^2\,u_1^5\,u_{1,1}^2\,u_{1,1,1,1,1,1,1}+1620\,u_{[\,]}^2\,u_1^5\,u_{1,1}\,u_{1,1,1}\,u_{1,1,1,1,1,1}+2820\,u_{[\,]}^2\\
&\quad u_1^5\,u_{1,1}\,u_{1,1,1,1}\,u_{1,1,1,1,1}+2204\,u_{[\,]}^2\,u_1^5\,u_{1,1,1}^2\,u_{1,1,1,1,1}+3028\,u_{[\,]}^2\,u_1^5\,u_{1,1,1}\,u_{1,1,1,1}^2-9\,u_{[\,]}^2\,u_1^4\,u_{1,1}^4-1536\,u_{[\,]}^2\\
&\quad u_1^4\,u_{1,1}^3\,u_{1,1,1,1,1,1}-12508\,u_{[\,]}^2\,u_1^4\,u_{1,1}^2\,u_{1,1,1}\,u_{1,1,1,1,1}-8548\,u_{[\,]}^2\,u_1^4\,u_{1,1}^2\,u_{1,1,1,1}^2-26624\,u_{[\,]}^2\,u_1^4\,u_{1,1}\\
&\quad u_{1,1,1}^2\,u_{1,1,1,1}-3432\,u_{[\,]}^2\,u_1^4\,u_{1,1,1}^4+8064\,u_{[\,]}^2\,u_1^3\,u_{1,1}^4\,u_{1,1,1,1,1}+68392\,u_{[\,]}^2\,u_1^3\,u_{1,1}^3\,u_{1,1,1}\,u_{1,1,1,1}+53004\,u_{[\,]}^2\\
&\quad u_1^3\,u_{1,1}^2\,u_{1,1,1}^3-33408\,u_{[\,]}^2\,u_1^2\,u_{1,1}^5\,u_{1,1,1,1}-129300\,u_{[\,]}^2\,u_1^2\,u_{1,1}^4\,u_{1,1,1}^2+101808\,u_{[\,]}^2\,u_1\,u_{1,1}^6\,u_{1,1,1}-25200\,u_{[\,]}^2\,u_{1,1}^8\\
&\quad -4\,u_{[\,]}\,u_1^8\,u_{1,1,1,1}-8\,u_{[\,]}\,u_1^8\,u_{1,1,1,1,1,1,1,1}+12\,u_{[\,]}\,u_1^7\,u_{1,1}\,u_{1,1,1}+56\,u_{[\,]}\,u_1^7\,u_{1,1}\,u_{1,1,1,1,1,1,1}+208\,u_{[\,]}\\
&\quad u_1^7\,u_{1,1,1}\,u_{1,1,1,1,1,1}+360\,u_{[\,]}\,u_1^7\,u_{1,1,1,1}\,u_{1,1,1,1,1}-6\,u_{[\,]}\,u_1^6\,u_{1,1}^3-368\,u_{[\,]}\,u_1^6\,u_{1,1}^2\,u_{1,1,1,1,1,1}-2088\,u_{[\,]}\\
&\quad u_1^6\,u_{1,1}\,u_{1,1,1}\,u_{1,1,1,1,1}-1416\,u_{[\,]}\,u_1^6\,u_{1,1}\,u_{1,1,1,1}^2-2320\,u_{[\,]}\,u_1^6\,u_{1,1,1}^2\,u_{1,1,1,1}+1968\,u_{[\,]}\,u_1^5\,u_{1,1}^3\,u_{1,1,1,1,1}\\
&\quad +12848\,u_{[\,]}\,u_1^5\,u_{1,1}^2\,u_{1,1,1}\,u_{1,1,1,1}+6864\,u_{[\,]}\,u_1^5\,u_{1,1}\,u_{1,1,1}^3-8256\,u_{[\,]}\,u_1^4\,u_{1,1}^4\,u_{1,1,1,1}-26184\,u_{[\,]}\,u_1^4\,u_{1,1}^3\,u_{1,1,1}^2\\
&\quad +25536\,u_{[\,]}\,u_1^3\,u_{1,1}^5\,u_{1,1,1}-7200\,u_{[\,]}\,u_1^2\,u_{1,1}^7+4\,u_1^9\,u_{1,1,1}+8\,u_1^9\,u_{1,1,1,1,1,1,1}-6\,u_1^8\,u_{1,1}^2-56\\
&\quad u_1^8\,u_{1,1}\,u_{1,1,1,1,1,1}-152\,u_1^8\,u_{1,1,1}\,u_{1,1,1,1,1}-104\,u_1^8\,u_{1,1,1,1}^2+312\,u_1^7\,u_{1,1}^2\,u_{1,1,1,1,1}+1312\\
&\quad u_1^7\,u_{1,1}\,u_{1,1,1}\,u_{1,1,1,1}+336\,u_1^7\,u_{1,1,1}^3-1344\,u_1^6\,u_{1,1}^3\,u_{1,1,1,1}-3096\,u_1^6\,u_{1,1}^2\,u_{1,1,1}^2+4224\,u_1^5\,u_{1,1}^4\,u_{1,1,1}-1440\\
&\quad u_1^4\,u_{1,1}^6\Big)
\end{aligned}
$$

We can use the Lie group action to cut down the expression swell. We can use the power of the Lie group based moving frames to derive Euler–Lagrange equations and conservation laws, directly using the invariant calculus, with links to Lie group integrators:
ELM, A practical guide to the invariant calculus, Cambridge Univ., Press., 2010.
T.M.N. Gonçalves and ELM, Moving frames and Noether's conservation laws – the general case. Forum of Math., Sigma, (2016).

Results in terms of a trivariational complex: I. Kogan and P.J. Olver, Acta Appl. Math **76** (2003)

From mathematical wallpaper to structure

Extend the projective $SL(2)$ action to include a dummy variable

$$g \cdot x = x, \qquad g \cdot t = t, \qquad g \cdot u = \frac{au + b}{cu + d}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

Again, via the chain rule, induce an action on u_t, u_{xt}, u_{xxt} ...

$$g \cdot u_t = \frac{\partial(g \cdot u)}{\partial(g \cdot t)} = \frac{u_t}{(cu + d)^2}$$

Same symbolic result from either of:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{L}[u + \epsilon v] \quad \leftrightarrow \quad \frac{\partial}{\partial t}\mathcal{L}[u], \qquad v \leftrightarrow u_t$$

Lowest order invariants are now

$$W = \frac{u_t}{u_x}, \qquad V = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} := \{u; x\}.$$

W is the invariantised variation and we need vary only in the direction of invariants.

V and W are functionally independent, but there is a differential identity or syzygy, in this case

$$\frac{\partial}{\partial t}V = \underbrace{(\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + V_x)}_{\mathcal{H}} W.$$

The syzygy plays a key role in finding the Euler Lagrange equations directly in terms of the invariants, for a Lie group invariant Lagrangian.

$$0 = \frac{\partial}{\partial t} \int L(x, V, V_x, V_{xx}, \ldots)\, \mathrm{d}x$$

$$= \int \left(\frac{\partial L}{\partial V} + \frac{\partial L}{\partial V_x} \frac{\partial}{\partial x} + \cdots \right) \frac{\partial}{\partial t} V \, \mathrm{d}x$$

$$= \int \underbrace{\left(\frac{\partial L}{\partial V} - \frac{\partial}{\partial x} \frac{\partial L}{\partial V_x} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial V_{xx}} + \cdots \right)}_{E^V(L)} \mathcal{H}(W)\, \mathrm{d}x + \text{ B.T's}$$

$$= \int \mathcal{H}^* \left(E^V(L) \right) W \, \mathrm{d}x + \text{ more B.T's}$$

where $\mathcal{H}^*$ is the adjoint of $\mathcal{H}$. Thus in this case,

$$E^u(L) = 0 \iff \mathcal{H}^* \left(E^V(L) \right) = 0.$$

and for this syzygy operator, it just so happens that $\mathcal{H}^* = -\mathcal{H}$.

For Lagrangians of the form $\int L(V, V_x, \dots)\,\mathrm{d}x$ where $V = \{u; x\}$, the laws can be written as

$$\mathbf{c} = \underbrace{\begin{pmatrix} a^2 & -ac & -c^2 \\ -2ab & ad+bc & 2dc \\ -b^2 & bd & d^2 \end{pmatrix}}_{R(g)^{-1}} \Bigg|_{g=\rho} \begin{pmatrix} \frac{\partial^2}{\partial x^2} E^V(L) + V E^V(L) \\ -2\frac{\partial}{\partial x} E^V(L) \\ -2E^V(L) \end{pmatrix}$$

where

$$\rho: \quad a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2(u_x)^{3/2}}, \qquad ad - bc = 1.$$

- $R(gh) = R(g)R(h)$, and $R(\rho(u, u_x, u_{xx}))$ is equivariant

Which representation yields $R(g)$? How to find ρ? And how to calculate the vector of invariants directly?

Answers and observations:

$$\begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \underbrace{\begin{pmatrix} a^2 & -ac & -c^2 \\ -2ab & ad+bc & 2dc \\ -b^2 & bd & d^2 \end{pmatrix}}_{R(g)^{-1}} \Bigg|_{g=\rho} \begin{pmatrix} \frac{\partial^2}{\partial x^2} E^V(L) + V E^V(L) \\ -2\frac{\partial}{\partial x} E^V(L) \\ -2E^V(L) \end{pmatrix}$$

$$\rho: \qquad a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2(u_x)^{3/2}}, \qquad ad - bc = 1.$$

- $R(g)$ is the (right) Adjoint representation of $SL(2)$
- We have three equations for u, u_x and u_{xx}. Writing the vector of invariants as $(v^1, v^2, v^3)^T$ and simplifying yields

$$\begin{aligned} 4c^1c^3 + (c^2)^2 &= 4v^1v^3 + (v^2)^2 \\ v^3 u_x &= -c^1u^2 + c^2u + c^3 \end{aligned}$$

A moving frame is an *equivariant* map $\rho : M \to G$ where

• in our case, M is the jet space with coordinates $(x, u, u_x, u_{xx}, \dots)$

• G is the Lie group, $SL(2)$

• and *equivariant* means,

$$\rho(g\cdot x, g \cdot u, g \cdot u_x, g \cdot u_{xx}, \dots) = \rho(x, u, u_x, u_{xx}, \dots)g^{-1}.$$

Noether's Theorem has calculated a moving frame without knowing any theory of such things.

But we do know about the theory of such things, and can use it to prove* theorems!

For smooth variational problems in dimension p, we obtain the conservation laws in terms of the Adjoint representation of a moving frame, p invariant vectors and p invariant 1-forms which are easy to calculate with, symbolically.

Simplest expression, with no group action on the base space:

$$\sum_i \frac{\mathrm{d}}{\mathrm{d}x_i} \mathcal{A}d(\rho)^{-1} v_i(I) = 0$$

where the v_i are known, once you have solved the Euler Lagrange system for the invariants.

*T.M.N. Gonçalves and E.L. Mansfield, *Moving Frames and Noether's Conservation Laws – the General Case*, Forum of Mathematics, Sigma **4** (2016) DOI: https://doi.org/10.1017/fms.2016.24

Strong use is made of the Fels and Olver[†] rewrite of Cartan's moving frame method, subsequently developed by Hubert, Kogan, and other authors, and as detailed in my book[‡].

Moving frames can be used to describe complete, or generating, sets of invariants and their relations.

There are excellent algorithms to manipulate quantities derived from moving frames in symbolic computation environments.

Moving frames are flexible, to allow for ease of computation in specific applications, and they satisfy equations that allow them to be obtained numerically (if necessary).

[†]Fels and Olver, Acta App. Math **51** (1998) and **55** (1999)

[‡]E.L. Mansfield, *A practical guide to the invariant calculus*, Cambridge Monographs on Applied and Computational Mathematics Volume 26, Cambridge University Press, Cambridge, 2010.

Smooth versus Finite Difference Calculus of Variations[§]
Basic Step

SMOOTH

$$\int_a^b f g_x \,\mathrm{d}x = -\int_a^b f_x g \,\mathrm{d}x + fg]_a^b$$

The L_2 adjoint of $\frac{\mathrm{d}}{\mathrm{d}x}$ is $-\frac{\mathrm{d}}{\mathrm{d}x}$.

The operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is a derivation, i.e. the product rule.

FIN. DIFF.

$$\sum f_n g_{n+1} = \sum f_{n-1} g_n + \underbrace{\sum (S - \mathrm{id})(f_{n-1} g_n)}_{\text{telescoping sum}}$$

The ℓ_2 adjoint of the Shift map S is S^{-1}.

The operator S is a homomorphism, $S(fg) = S(f)S(g)$.

The operator $S - id$ is the required total difference operator for a conservation law, but is otherwise useless.

[§]Kuperschmidt; Hydon and Mansfield, FoCM 2004

For smooth $\mathcal{L}[u] = \int L(x, u, u_x, \ldots, u_{Nx})\,\mathsf{d}x$, the Euler–Lagrange equation is

$$E^u(L) = \sum \left(-\frac{\mathsf{d}}{\mathsf{d}x}\right)^j \frac{\partial L}{\partial u_{jx}} = 0.$$

For finite difference $\mathcal{L}[u] = \sum L(n, u_n, u_{n+1}, \ldots, u_{n+N})$, the Euler–Lagrange equation is

$$E^u_\Delta(L) = \sum S^{-j} \frac{\partial L}{\partial u_{n+j}} = 0.$$

The boundary terms which give rise to the Noether laws have the same kind of relationship. **Many** authors have discovered finite difference Noether laws. Most general differential-difference results by L. Peng, Studies Applied Math.

Simplest Example[¶]

The finite difference approximation of $\int \left(\frac{1}{2}x_t^2 + V(x)\right) \mathrm{d}t$ is

$$\sum \left[\frac{1}{2}\left(\frac{x_{n+1}-x_n}{t_{n+1}-t_n}\right)^2 + V(x_n)\right](t_{n+1}-t_n)$$

$$\begin{aligned}
0 &= E^x_\Delta(L) = \frac{\partial L}{\partial x_n} + S^{-1}\frac{\partial L}{\partial x_{n+1}} \\
&= (S^{-1}-\mathrm{id})\left(\frac{x_{n+1}-x_n}{t_{n+1}-t_n}\right) + \frac{\mathrm{d}V}{\mathrm{d}x_n}(t_{n+1}-t_n) \\
0 &= E^t_\Delta(L) = \frac{\partial L}{\partial t_n} + S^{-1}\frac{\partial L}{\partial t_{n+1}} = (S-\mathrm{id})\left(-S^{-1}\frac{\partial L}{\partial t_{n+1}}\right) \\
&\Longrightarrow -\frac{1}{2}\left(\frac{x_{n+1}-x_n}{t_{n+1}-t_n}\right)^2 + V(x_n) = \text{constant}
\end{aligned}$$

The constant of integration is due to invariance under translation in $t_n \mapsto t_n + \epsilon$. Also $\exists$ a conserved symplectic form!

[¶] T.D Lee, 1987, J. Stat. Phys., Introduction.

For difference variational methods, we have similar results on the difference Noether Theorem[‖] ,

$$0 = \sum_i (S_i - \mathrm{id}) \mathcal{A}d(\rho_0)^{-1} v_0^i(I) = 0.$$

This time we use a discrete moving frame, as developed by myself and Gloria Marí Beffa[**].

[‖] E.L. Mansfield, A. Rojo–Echeburúa, L. Peng and P.E. Hydon, *Moving Frames and Noether's Finite Difference Conservation Laws I, II*, Trans. Math. App.

[**] E.L. Mansfield, G. Marí Beffa, J.P. Wang, *Discrete moving frames and applications.*, Foundations of Computational Mathematics, **13**, 545–582, (2013), and G. Marí Beffa and E.L. Mansfield, *Discrete moving frames on lattice varieties and lattice based multispace*, Foundations of Computational Mathematics **18**, 181–247, (2018).

A discrete frame is essentially a sequence of frames.

A difference frame is a discrete frame with $\rho_{n+1} = S\rho_n$.

	Smooth**	Difference
Invariants	$\mathcal{Q}^x = \rho_x \rho^{-1}$ $\mathcal{Q}^t = \rho_t \rho^{-1}$	$K_n = \rho_{n+1}\rho_n^{-1}$ $N_n^t = \rho_{n,t}\rho_n^{-1}$
Syzygies	$\partial_t \mathcal{Q}^x - \partial_x \mathcal{Q}^t = \left[Q^t, Q^x\right]$	$\partial_t K_n = S(N_n^t)K_n - K_n N_n^t$

** Given here for invariant independent variables.

The significant examples considered include

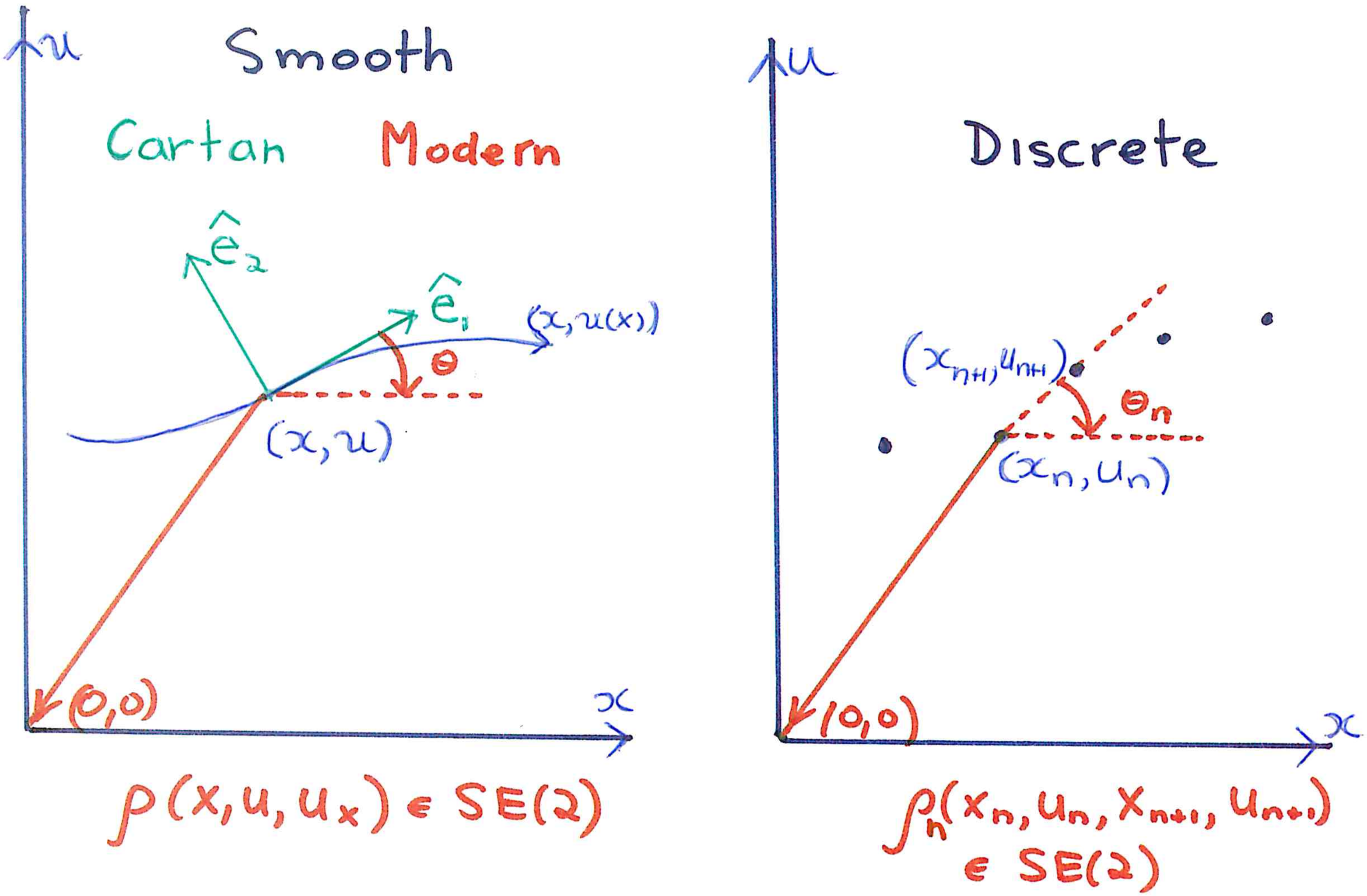
• a difference version of the $SE(2)$ invariant Lagrangian for Euler's Elastica,

$$\mathcal{L}[u] = \int \kappa^2 \, \mathrm{d}s$$

where κ is the Euclidean curvature of a curve $(x, u(x))$ and s is the Euclidean arclength.

The example concerns having a difference model of the smooth system which is symplectic and which has all three conservation laws built in, in the sense that the discrete Euler Lagrange equations and the laws all have the relevant smooth equations and laws as a continuum limit.

A simple method for this simple Lagrangian: Step 1: match the smooth and the discrete frames, to first order.


Smooth
Cartan
Modern
ê2
ê1
(x, u(x))
θ
(x, u)
(0,0)
u
x
ρ(x, u, ux) ∈ SE(2)
Discrete
(xn+1, un+1)
θn
(xn, un)
(0,0)
u
x
ρn(xn, un, xn+1, un+1)
∈ SE(2)

Results for the smooth Euler Elastica case[††]

$\mathcal{L}[u] = \int \kappa^2 \,\mathrm{d}s, \qquad \kappa = u_{xx}/(1+u_x^2)^{3/2}, \quad \mathrm{d}s = (1+u_x^2)^{1/2}\mathrm{d}x$

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} x_s & u_s & 0 \\ -u_s & x_s & 0 \\ u & -x & 1 \end{pmatrix}^{-1}}_{\mathcal{A}d(\rho)^{-1}} \begin{pmatrix} -\kappa^2 \\ -2\kappa_s \\ 2\kappa \end{pmatrix}$$

[††]T.M.N. Gonçalves and E.L. Mansfield, *Moving Frames and Conservation Laws for Euclidean Invariant Lagrangians*, Studies in Applied Mathematics **130** (2013), 134–166.

Putting $\mathcal{A}d(\rho)$ to the other side, we have,

$$\begin{pmatrix} x_s & u_s & 0 \\ -u_s & x_s & 0 \\ u & -x & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -\kappa^2 \\ -2\kappa_s \\ 2\kappa \end{pmatrix}$$

We see clearly a first integral for the Euler Lagrange equation,

$$c_1^2 + c_2^2 = \kappa^4 + 4\kappa_s^2,$$

a linear relation between u and x, and a single remaining ordinary differential equation to solve,

$$x_s = \frac{1}{c_1^2 + c_2^2} \left(c_1 \kappa^2 + 2 c_2 \kappa_s \right).$$

Two representations of $SE(2)$ are

Standard	(Right) Adjoint
$g(\theta, a, b) =$	$\mathcal{A}d(g(\theta, a, b)) =$

$$\begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ a\sin\theta - b\cos\theta & b\sin\theta + a\cos\theta & 1 \end{pmatrix}$$

The frames are, dropping the n's in the indices,

$$\mathcal{A}d(\rho(s)) = \begin{pmatrix} x_s & u_s & 0 \\ -u_s & x_s & 0 \\ u & -x & 1 \end{pmatrix}$$

$$\mathcal{A}d(\rho_0) = \begin{pmatrix} \frac{x_1 - x_0}{\ell_0} & \frac{u_1 - u_0}{\ell_0} & 0 \\ -\frac{u_1 - u_0}{\ell_0} & \frac{x_1 - x_0}{\ell_0} & 0 \\ u_0 & -x_0 & 1 \end{pmatrix}$$

so that

$$\tan\theta(s) = -\frac{u_s}{x_s}$$

$$x_s^2 + u_s^2 = 1$$

$$\tan\theta_0 = -\frac{u_1 - u_0}{x_1 - x_0}$$

$$\ell_0 = \sqrt{(x_1 - x_0)^2 + (u_1 - u_0)^2}.$$

We have in the Adjoint representation for $SE(2)$ that the Maurer–Cartan matrices are

$$K := \mathcal{A}d(\rho)_s \mathcal{A}d(\rho)^{-1} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$K_0 := \mathcal{A}d(\rho_1)\mathcal{A}d(\rho_0)^{-1} = \begin{pmatrix} R_{\Delta\theta_0} & & 0 \\ 0 & -\ell_0 & 1 \end{pmatrix}$$

where

$$R_{\Delta\theta_0} = \begin{pmatrix} \cos\Delta\theta_0 & -\sin\Delta\theta_0 \\ \sin\Delta\theta_0 & \cos\Delta\theta_0 \end{pmatrix}, \quad \ell_0 = \sqrt{(x_1 - x_0)^2 + (u_1 - u_0)^2}.$$

Step 2: Now think: $\rho_1\rho_0^{-1} \approx \mathrm{Id} + (\delta s)\rho_s\rho^{-1}|_{(x_0,u_0)}$.

So, we took a first order approximation to be

$$\kappa \;\leftrightarrow\; -\frac{\sin\Delta\theta_0}{\ell_0}$$

$$\mathrm{d}s \;\leftrightarrow\; \ell_0$$

and we considered the approximation of $\int \kappa^2\,\mathrm{d}s$ to be

$$\sum \frac{(\sin\Delta\theta_0)^2}{\ell_0}.$$

Using the methodology and Theorems in our paper, we arrived at Euler Lagrange equations for the invariants $(\Delta\theta_r)$ and (ℓ_r), and three conservation laws.

The difference laws look like $\mathbf{c} = \mathcal{A}d(\rho_0)^{-1}\mathbf{v}_0$ or

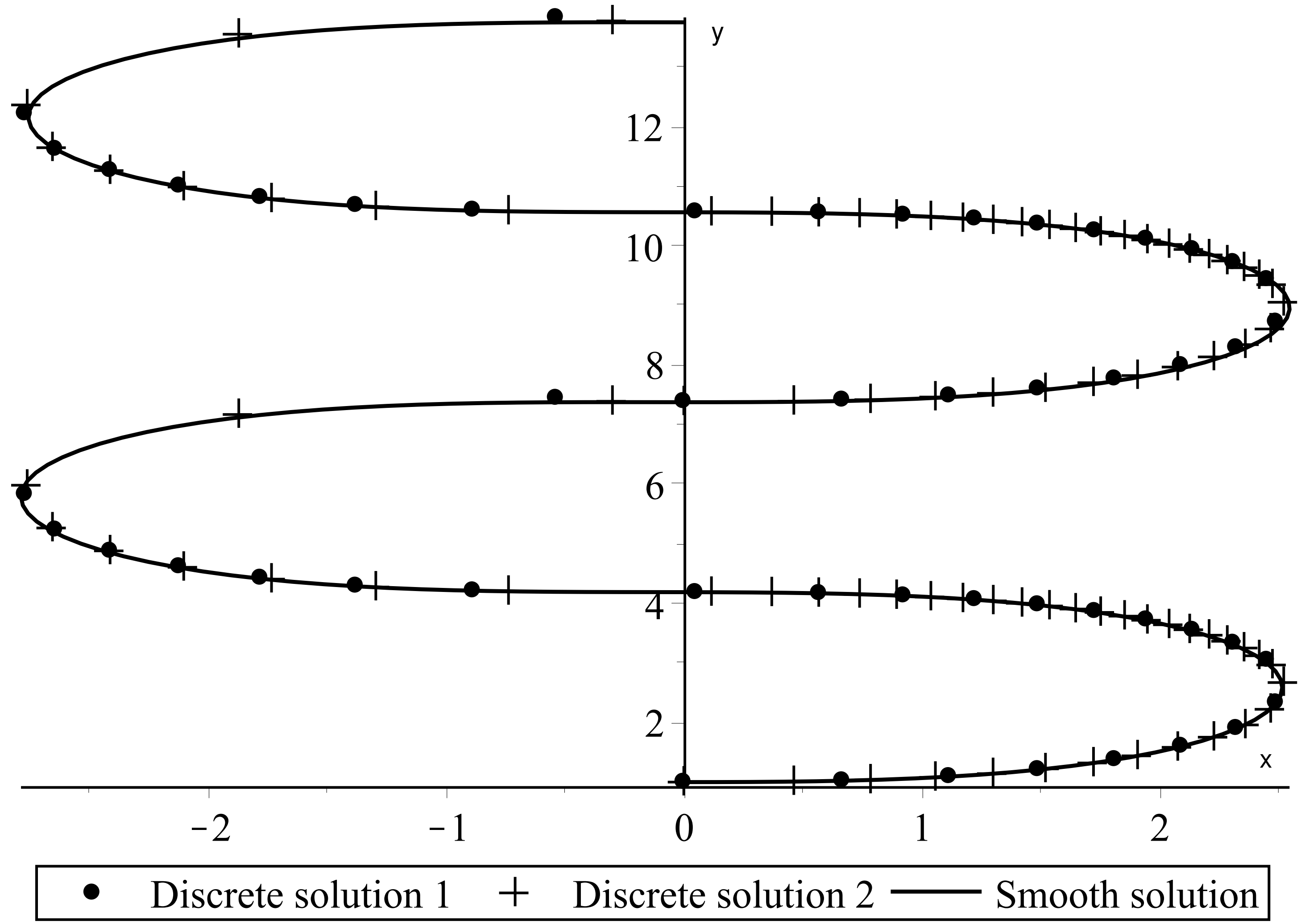
$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 \\ \sin\theta_0 & \cos\theta_0 & 0 \\ u_0 & -x_0 & 1 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \begin{pmatrix} v_0^1 \\ v_0^2 \\ v_0^3 \end{pmatrix}$$

where the (v_r^i) are known functions of the $(\Delta\theta_r)$ and the (ℓ_r), and which are known in terms of n, once the Euler Lagrange equations have been solved. We can see

$$(c^1)^2 + (c^2)^2 = (v_0^1)^2 + (v_0^2)^2, \qquad \tan\theta_0 = \frac{c^1 v_0^2 - c^2 v_0^1}{c^1 v_0^1 + c^2 v_0^2},$$

and a linear relation for x_0 and u_0.

Initial data: use $\rho_1 = K_0(\Delta\theta_0, \ell_0)\rho_0$ and $\binom{x_0}{u_0} = \rho_0^{-1}\binom{0}{0}$.



y
12
10
8
6
4
2
x
−2
−1
0
1
2
Discrete solution 1
Discrete solution 2
Smooth solution

What price the Ge and Marsden theorem?

Example Consider these two Lagrangians, one smooth, one finite difference, their conservation laws and their conserved symplectic forms:

$\mathcal{L} = \int L(u, u_t)\,\mathrm{d}t$	$\widetilde{\mathcal{L}} = \sum \widetilde{L} = \sum L\left(u_n, \frac{u_{n+1}-u_n}{t_{n+1}-t_n}\right)(t_{n+1} - t_n)$
$c = L - u_x D_2(L)$	$c = L - \frac{u_{n+1}-u_n}{t_{n+1}-t_n} D_2(L)$
$\mathrm{d}u \wedge \mathrm{d}\,(D_2(L))$	$\mathrm{d}u_{n+1} \wedge \mathrm{d}\frac{\partial \widetilde{L}}{\partial u_{n+1}} + \mathrm{d}t_{n+1} \wedge \mathrm{d}\frac{\partial \widetilde{L}}{\partial t_{n+1}}$

To incorporate the physics into the numerical model, need to avoid the Ge and Marsden "no go" theorem, so:

♢ make the discrete Lagrangian to be

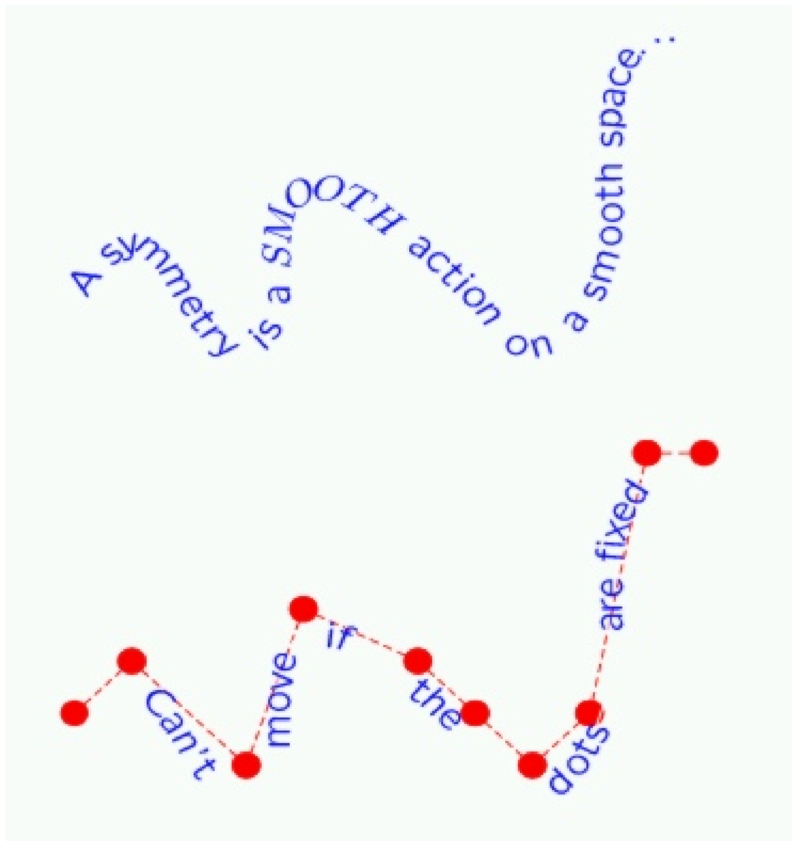
♡ *invariant under the induced action on the approximation data*, and

♡ have the correct continuum limit

♢ write down the exactly conserved (in approximation space), discrete Noether law

♢ prove the discrete Euler–Lagrange equation and the discrete conservation laws, converge to the desired smooth equations and laws in some useful sense.

When constructing a discrete Noether's theorem for your approximation model, the big challenge is to find where the group action has gone to!



For Finite Difference methods, where the approximation data is the value at a point, you have to have the coordinates of the independent variables as new dependent variables, whose values are referred to a fixed (dummy) grid.

For Finite Elements, where the approximation data takes the form of average values over edges and faces, we can induce actions as follows,

$$\int_\sigma f(x,u)\,\mathrm{d}x \mapsto \int_\sigma f(g\cdot x, g\cdot u)\frac{\partial(g\cdot x)}{\partial x}\,\mathrm{d}x.$$

ELM and Pryer: Noether-type Discrete Conserved Quantities arising from a Finite Element approximation of a variational problem, FoCM, **17** (3) 2017.

An earlier version of mine using D. Arnold's complexes was never tested. Proc. FoCM, 2005.

Recall the link between extremisation and Noether's laws starts with:

$$\underbrace{0 =}_{\text{at extremal}} \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \int L(x, u + \epsilon v, u_x + \epsilon v_x, \ldots, u_{(nx)} + \epsilon v_{(nx)})\,\mathrm{d}x$$

versus

$$\underbrace{0 =}_{\text{invariance}} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int L(g(t)\cdot x, g(t)\cdot u, g(t)\cdot u_x, \ldots, g(t)\cdot u_{(nx)}) \frac{\mathrm{d}(g(t)\cdot x)}{\mathrm{d}x}\,\mathrm{d}x$$

with $g(t) \subset G$ and $g(0) = e$, the identity element.

And we take this to be the the starting point for the discrete Noether's Theorem. If $\mathbf{p}$ is the approximation data, we have

$$\underbrace{0 =}_{\text{at extremal}} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \int \bar{L}(p_1 + \epsilon v_1, p_2 + \epsilon v_2, \ldots, p_n + \epsilon v_n)\,\overline{\mathrm{d}x}$$

and

$$\underbrace{0 =}_{\text{invariance}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int \bar{L}(g(t)\cdot p_1, g(t)\cdot p_2, g(t)\cdot p_3, \ldots, g(t)\cdot p_n)\, g(t)\cdot\overline{\mathrm{d}x}$$

with $\bar{L}$ the approximate Lagrangian and $\overline{\mathrm{d}x}$ the approximate volume form.

Result for FEM: The relevant Noether's theorem will give an exact conservation law of the approximate problem.

- The weaker the invariance in the functional analytic sense, the weaker the law.

- No symmetry conditions on the mesh are required.

- For symmetries that are linear actions on the base space, ordinary Lagrangian elements can be used.

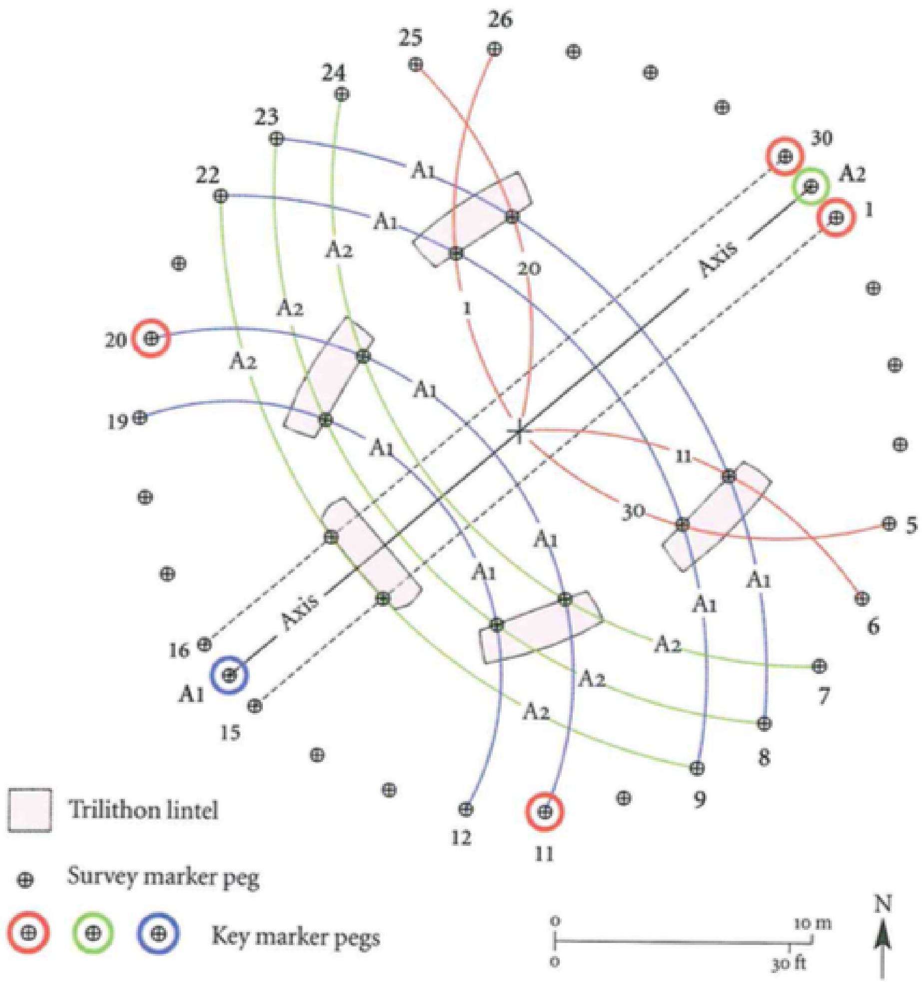
- Because integration by parts is only valid piecewise in the relevant function spaces, the laws have a different look and feel.

A bit of fun: in which we show evolving "sound" waves of a drum beating in the heart of Stonehenge

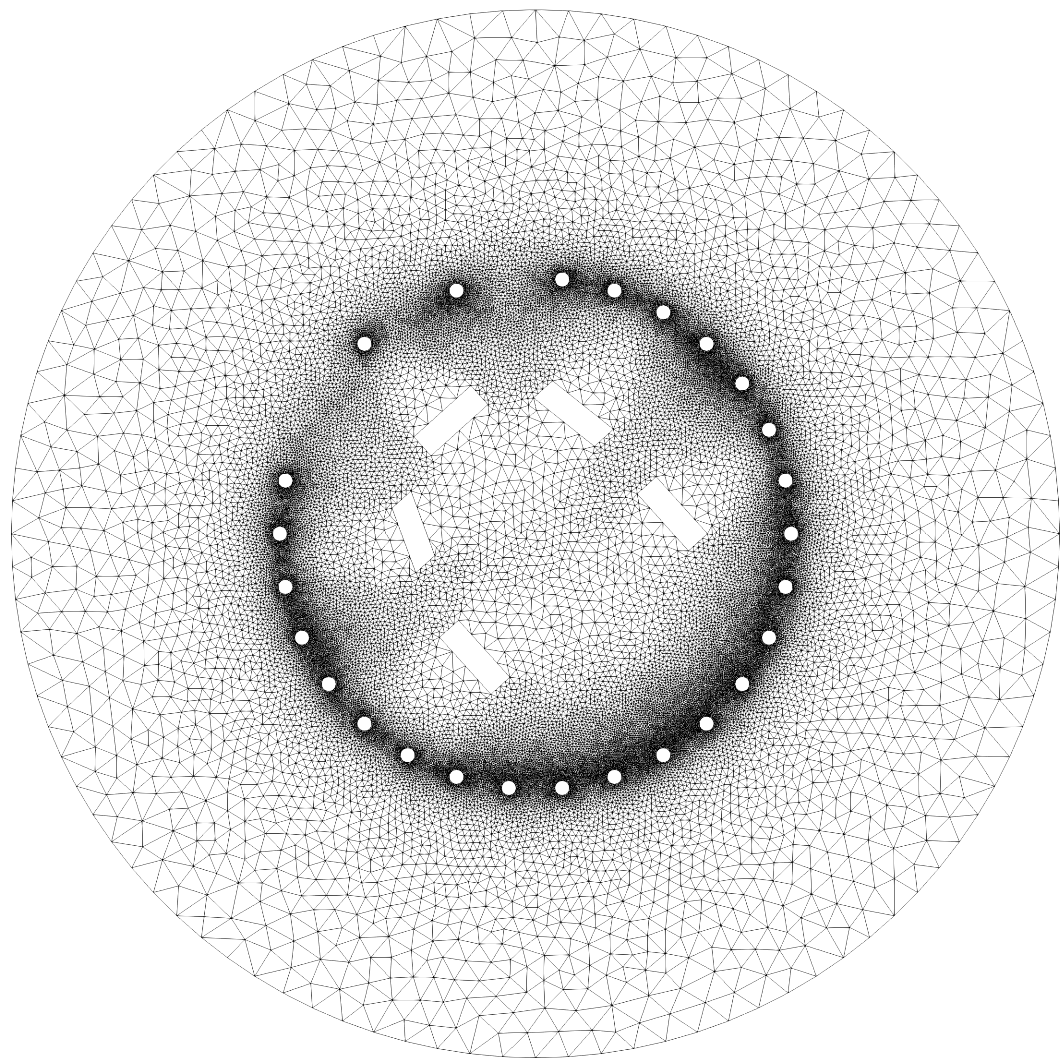
• shallow water-type equations and FEM

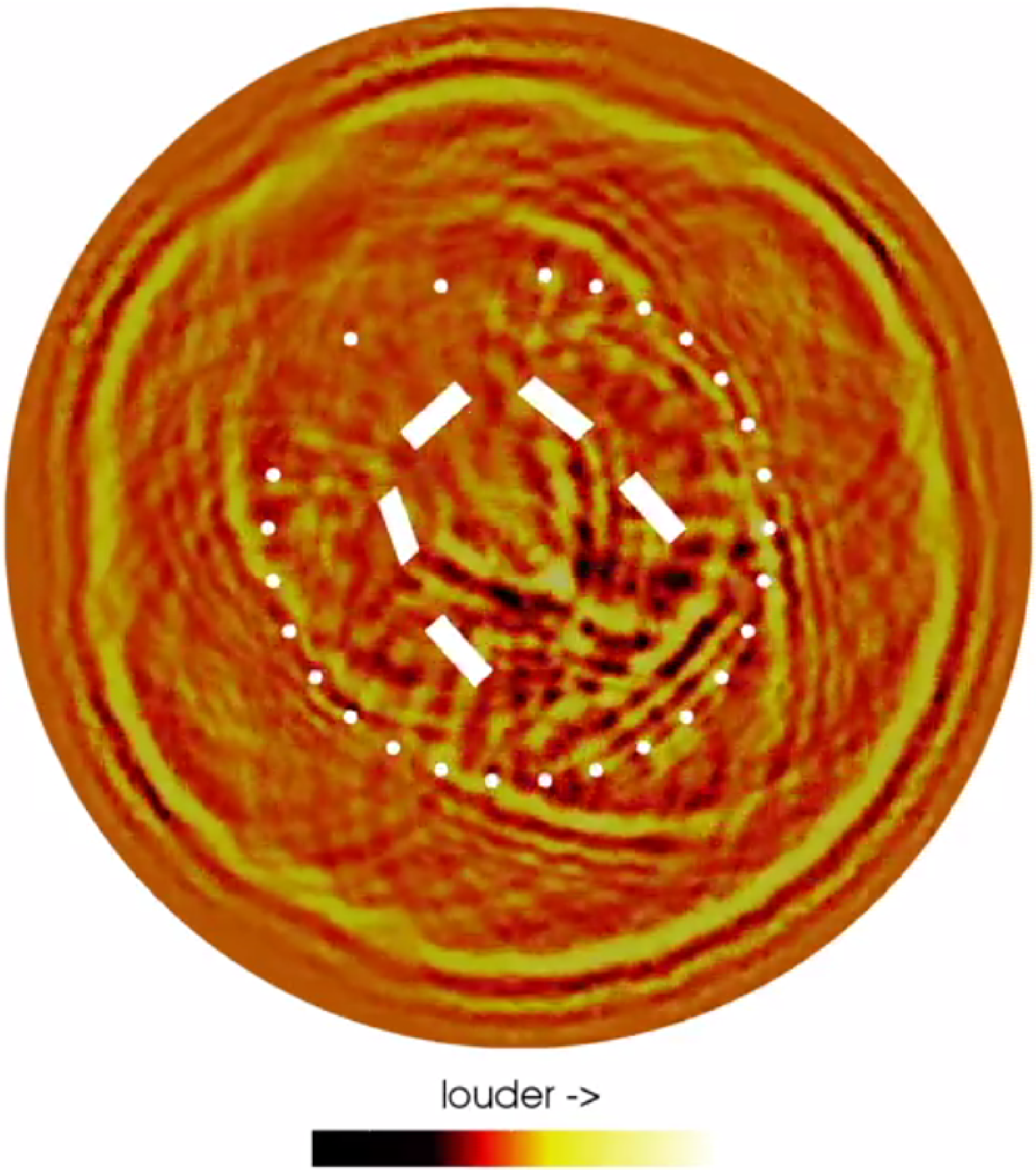
• exact energy conservation using a trick from geometric integration of ODES called "the discrete gradient method"
• weak conservation of linear and angular momentum, à la ELM and Pryer.

We use a precise survey of Stonehenge, using only pegs and ropes, discovered by Anthony Johnson, "Solving Stonehenge: the new key to an ancient enigma", Thames & Hudson, 2008.



25
26
24
23
22
30
A2
1
A1
A2
20
19
Axis
11
30
5
6
16
A1
15
7
8
9
12
11
Trilithon lintel
Survey marker peg
Key marker pegs
0
10 m
0
30 ft
N




louder ->

Also available: an extension of the smooth Noether's Second Theorem and its finite difference analogue:

P.E. Hydon and ELM, (2011) Extensions of Noether's Second Theorem: from continuous to discrete systems, *Proc. Roy. Soc., Lond. A* **467**:3206–3221.

FEM-style conservation of potential vorticity is also proved theoretically, but needs a numerical experiment to complete the project.

THANK YOU!!