Rigidity problems for the lengths of geodesics in Riemannian geometry

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Introduction

We will discuss two types of (related) rigidity problems:

1) Riemannian invariants: marked length spectrum rigidity
2) Inverse problems/tomography: boundary/lens rigidity problem

Kac Problem

- (M, g) closed Riemannian manifold (possibly with boundary)
- $\mathrm{Sp}(\Delta_g) = \{\lambda_i \geq 0; \ker(\Delta_g - \lambda_i) \neq 0\}$ the Laplace spectrum

Kac problem: Does $\mathrm{Sp}(\Delta_g)$ determine g up to isometry?

Before I go any further, let me say that as far as I know the problem is still unsolved. Personally, I believe that one cannot "hear" the shape of a tambourine but I may well be wrong and I am not prepared to bet large sums either way.

Answer: No in general, $\exists$ counter examples:

Milnor '64 - flat torii with dim $= 16$
Vignéras '80, Sunada '85: hyperbolic closed manifolds
Gordon-Webb-Wolpert '92: domains with (non-convex, non smooth) boundary

Positive results:

- The disc in $\mathbb{R}^2$ is spectrally rigid (Kac '66)
- $\nexists$ 1−parameter family of isospectral metrics with $K_g \leq 0$ (Guillemin-Kazhdan '80, Croke-Sharafutdinov '98, Paternain-Salo-Uhlmann '13)
- Compactness of isospectral sets in C^∞ (Melrose '83, Osgood-Phillips-Sarnak '88, Brooks-Perry-Petersen '92) in dim 2 and 3.
- Ellipses with small eccentricity are spectrally rigid (Hezari-Zelditch '19)

Tools:

- Heat trace invariants: $\mathrm{Tr}(e^{-t\Delta_g}) = \sum_j e^{-t\lambda_j}$ as $t \to 0$),
- $\det(\Delta_g)$,
- singularities of wave trace $\mathrm{Tr}(e^{-it\sqrt{\Delta_g}}) = \sum_j e^{-it\sqrt{\lambda_j}}$ in $t > 0$.

Isospectral local rigidity (negative curvature)

Osgood-Phillips-Sarnak problem:

If g and g_0 are close enough with $K_{g_0} < 0$ and $\mathrm{Sp}(\Delta_g) = \mathrm{Sp}(\Delta_{g_0})$, then g isometric to g_0?

Consequences: That would imply finiteness of isospectral sets (up to isometry)

Positive result: True if g_0 satisfies $K_{g_0} = -1$ (Sharafutdinov '09).

Length spectrum rigidity

- (M, g) closed Riemannian manifold with $K_g < 0$
- $\mathrm{LS}(g) = \{\ell_g(\gamma); \gamma \text{ closed geodesic}\}$ the length spectrum
- wave-trace singularities at $\mathrm{LS}(g)$ (Balian-Bloch '71, Colin de Verdière '73, Chazarain '74, Duistermaat-Guillemin '75):

$$(t - \ell_\gamma)\mathrm{Tr}(e^{-it\sqrt{\Delta_g}}) \sim \frac{1}{2\pi}\ell_\gamma^\sharp |\det(1 - P_\gamma)|^{-1/2}$$

Length spectrum rigidity problem: Does $\mathrm{LS}(g)$ determine g up to isometry?

Answer: No! counter examples as for $\mathrm{Sp}(\Delta_g)$

Length spectrum local rigidity: If g and g_0 are close enough and $\mathrm{LS}(g) = \mathrm{LS}(g_0)$, then g isometric to g_0?

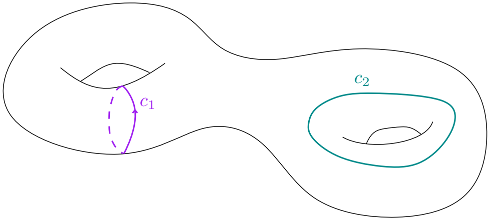
No known results.

Marked length spectrum rigidity

- (M, g) closed Riemannian manifold with $K_g < 0$
- $\mathcal{C} :=$ set of free homotopy classes on M
- each $c \in \mathcal{C}$ contains a unique closed geodesic γ_c
- marked length spectrum (= length spectrum with ordering):

$$L_g : \mathcal{C} \to \mathbb{R}^+, \qquad L_g(c) := \ell_g(\gamma_c)$$

Conjecture (Burns-Katok '85): $L_g = L_{g'}$ implies g isometric to g'.



The linearised marked length operator

- $\mathcal{G}$ = set of closed geodesics γ on $M \simeq \mathcal{C}$
- linearisation of $g \mapsto L_g / L_{g_0} \in L^\infty(\mathcal{G})$ at g_0: the X-ray transform on 2-tensors

$$I_2 : C^0(M; S^2 T^* M) \to L^\infty(\mathcal{G}),$$

$$I_2 f(\gamma) := \frac{1}{\ell_{g_0}(\gamma)} \int_0^{\ell_{g_0}(\gamma)} f_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt$$

Linearised problem:

- s-injectivity: $\ker I_2 = \{\mathcal{L}_V g_0; V \in C^1(M; TM)\}$?
- Stability estimates: $\|I_2 f\| \geq C \|f\|$ for $f \perp \ker I_2$?

Positive results (marked length spectrum rigidity)

Non-linear problem:

- dim 2: Otal '90, Croke '90
- dim $n > 2$ and g is conformal to g': Katok '88
- dim $n > 2$ when (M, g) is a locally symmetric space and $K_g < 0$, Besson-Courtois-Gallot '95, Hamenstädt '99

Linearised problem:

- s-injectivity of I_2 when $K_g < 0$: Guillemin-Kazhdan '80, Croke-Sharafutdinov '98
- dim 2: s-injectivity of I_2 when g has Anosov geodesic flow: Paternain-Salo-Uhlmann '14

Local rigidity of marked length spectrum

Theorem (G-Lefeuvre '18)

Let (M, g) be either

- *a closed surface with Anosov geodesic flow, or*
- *a closed manifold of dim $n > 2$ with $K_g \leq 0$ and Anosov geodesic flow.*

There is a C^k neighborhood U of g such that if $g' \in U$ and $L_g = L_{g'}$, then g' is isometric to g.

Thurston distance

Thurston distance: g_1, g_2 negatively curved metrics,

$$d_T(g_1, g_2) := \limsup_{j\to\infty} \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)}$$

Theorem (Thurston '98)

On Teichmüller space $\mathcal{T}_M := \{g \mid K_g = -1\}/\mathrm{Diff}_0(M)$, d_T is an asymmetric distance: $d_T(g_1, g_2) > 0$ unless $g_1 = g_2$.

Stability and Thurston distance

Theorem (G-Knieper-Lefeuvre '19)

Let (M, g_0) be as in previous theorem. Then $\exists k \in \mathbb{N}$, $\varepsilon > 0$ and $C_{g_0} > 0$ such that for all g_1, g_2 metrics such that $\|g_1 - g_0\|_{C^k} \leq \varepsilon$, $\|g_2 - g_0\|_{C^k} \leq \varepsilon$, there is a C^k- diffeomorphism $\psi : M \to M$ such that

$$\|\psi^* g_2 - g_1\|_{H^{-\frac{1}{2}}(M)} \leq C_{g_0} |d_T(g_1, g_2)|^{\frac{1}{2}}$$

In particular $L_{g_1} = L_{g_2}$ implies g_2 isometric to g_1, and d_T symmetrized defines a distance near the diagonal of

$$\{Isometry\ classes\} \times \{Isometry\ classes\}.$$

Remark: using interpolation, can be upgraded to

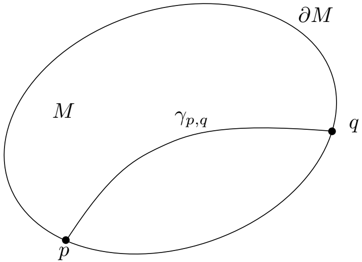
$$\|\psi^* g_1 - g_2\|_{C^{k'}} \leq C_{g_0} |d_T(g_1, g_2)|^{\delta}$$

for some $\delta > 0$ depending on $k' < k$.

Boundary rigidity problem (Michel conjecture)

- (M,g): smooth compact manifold with ∂M strictly convex
- $d_g : M \times M \to \mathbb{R}^+$ the Riemannian distance
- $\beta_g := d_g|_{\partial M\times\partial M}$ the restriction to ∂M

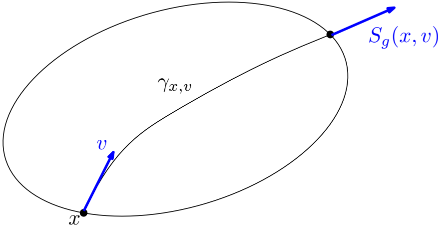
Boundary rigidity pb: does β_g determine g up to isometries fixing ∂M ?



Lens rigidity problem

- (M, g): smooth compact manifold with ∂M strictly convex
- $SM := \{(x, v) \in TM \,|\, g_x(v, v) = 1\}$
- $\varphi_t : SM \to SM$ geodesic flow
- for $(x, v) \in \partial SM$, let $\ell_g(x, v) :=$ length of geodesic $\gamma_{(x,v)}$
- for $(x, v) \in \partial SM$, let $S_g(x, v) := \varphi_{\ell_g(x,v)}(x, v)$ scattering map

Lens rigidity prb: does (ℓ_g, S_g) determine g up to isometries fixing ∂M ?



The linearised operator in the boundary case - X ray transform

We linearise the non-linear map $g \mapsto \beta_g$ at g_0:

- $\mathcal{G}$ =set of geodesics γ (for g_0) with endpoints on ∂M
- linearised operator: *X-ray transform* on 2-tensors:

$$I_2 : C^0(M; S^2T^*M) \to L^\infty_{\rm loc}(\mathcal{G}),$$

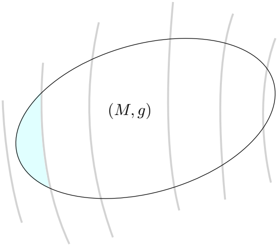
$$I_2 f(\gamma) := \int_\gamma f = \int_0^{\ell_{g_0}(\gamma)} f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt$$

Linearised pb:

- s-injectivity: $\ker I_2 = \{\mathcal{L}_V g_0 \,|\, V \in C^1(M; TM), V|_{\partial M} = 0\}$?
- Stability estimates: $\|I_2 f\| \geq C\|f\|$ for $f \perp \ker I_2$?

Positive results - boundary/lens rigidity

- dim 2: Otal ('90), Croke ('90) if $K_g \leq 0$ & simply connected.
  Pestov-Uhlmann ('03) if no conjugate points & simply connected (*simple metrics*).
- dim $n > 2$: Stefanov-Uhlmann-Vasy ('17) if strictly convex foliation. Satisfied if (M, g)=topological ball and $K_g \leq 0$

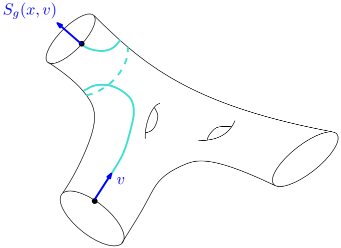


- s-Injectivity of I_2 with stability in cases above: Pestov-Sharafutdinov '88, Stefanov-Uhlmann-Vasy '14, Paternain-Salo-Uhlmann '13.

Our contribution - lens rigidity

Theorem (G '17)

On negatively curved surfaces with ∂M convex, S_g determines (M, g) up to conformal diffeomorphisms fixing ∂M. Moreover I_2 is s-injective with stability estimates.



Remark: First general result for non simply connected manifolds.

Main difficulty: some geodesics are trapped.

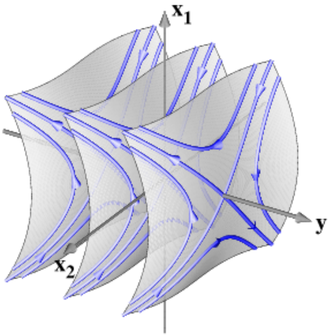
Axiom A and Anosov flows

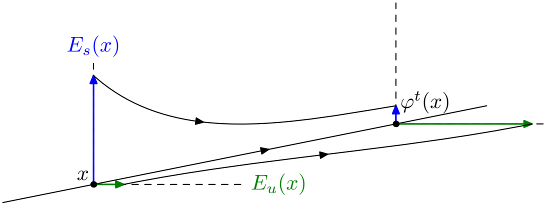
- $\mathcal{M}$ a smooth compact manifold with or without boundary
- X a smooth non-vanishing vector field on $\mathcal{M}$, with flow φ_t, such that $\partial\mathcal{M}$ is strictly convex for the flow lines of X (or $\partial\mathcal{M} = \emptyset$).
- $K := \cap_{t\in\mathbb{R}}\varphi_t(\mathcal{M}^\circ)$ the *trapped set*, is closed flow-invariant, contains the closed orbits ($K = \mathcal{M}$ if $\partial\mathcal{M} = \emptyset$)
- Assume K is hyperbolic for φ_t: i.e. flow-invariant splitting

$$\exists\nu > 0, \quad T_K\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u$$

$$\|d\varphi_t|_{E_s}\| \leq Ce^{-\nu t}, \ \forall t \gg 1, \quad \|d\varphi_t|_{E_u}\| \leq Ce^{-\nu|t|}, \ \forall t \ll -1$$

- if $\partial\mathcal{M} = \emptyset$, the flow is said Anosov





Examples: geodesic flow on $\mathcal{M} = SM$ if (M, g) has negative curvature and either ∂M strictly convex or empty.

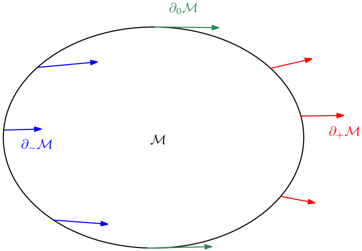
Analytic methods for the previous problems

- $\mathcal{M}$: smooth compact manifold with boundary
- X: smooth vector field on $\mathcal{M}$ with flow φ_t
- $\partial\mathcal{M}$ strictly convex for the flow lines of X (or $\partial\mathcal{M} = \emptyset$)

$$\partial_0\mathcal{M} = \{y \in \partial\mathcal{M}; X(y) \text{ tangent to } \partial\mathcal{M}\}$$

$$\partial_-\mathcal{M} = \{y \in \partial\mathcal{M}; X(y) \text{ pointing inside } \mathcal{M}\}$$

$$\partial_+\mathcal{M} = \{y \in \partial\mathcal{M}; X(y) \text{ pointing outside } \mathcal{M}\}$$



Boundary value problems, well-posedness

Let μ be a smooth measure invariant by φ_t, $V \in C^\infty(\mathcal{M})$ a potential.

Question: for $f \in C^\infty(\mathcal{M})$ (or $L^p(\mathcal{M}), H^s(\mathcal{M})$,...), can we solve the linear PDE (transport equation)

$$(X + V)u = f, \quad u|_{\partial_-\mathcal{M}} = 0$$

in a given functional space? Is the solution unique? singularities of u?

Remark:

- In $\mathcal{D}'(\mathcal{M})$, no uniqueness: if $V = 0$ and X has a periodic orbit γ not intersecting $\partial\mathcal{M}$, then $X\delta_\gamma = 0$.
- if $\partial\mathcal{M} = \emptyset$ and φ_t is ergodic, $\ker X \cap L^p(\mathcal{M}) = \mathbf{R}$

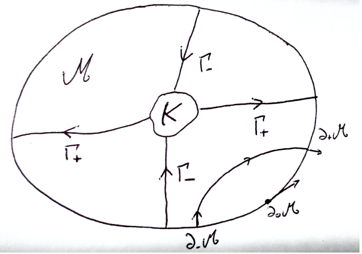
Trapped sets

Define the exit times from $\mathcal{M}$

$$\ell_+ : \mathcal{M} \to [0, \infty], \quad \ell_+(y) = \sup(\{t \geq 0; \varphi_t(y) \in \mathcal{M}^\circ\} \cup \{0\})$$
$$\ell_- : \mathcal{M} \to [-\infty, 0], \quad \ell_-(y) = \inf(\{t \leq 0; \varphi_t(y) \in \mathcal{M}^\circ\} \cup \{0\}).$$

Introduce the

- forward/backward trapped set $\Gamma_\mp := \{y \in \mathcal{M}; \ell_\pm = \pm\infty\}$,
- trapped set $K := \Gamma_- \cap \Gamma_+$



Resolvents (case $V = 0$)

Add a damping: let $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$ and define the operators

$$R_+(\lambda)f(y) = \int_0^{\ell_+(y)} e^{-\lambda t} f(\varphi_t(y))dt,$$

$$R_-(\lambda)f(y) = -\int_{\ell_-(y)}^0 e^{\lambda t} f(\varphi_t(y))dt.$$

bounded on $L^2(\mathcal{M}, \mu)$. They solve the boundary value pb

$$\begin{cases} (-X - \lambda)R_-(\lambda)f = f \\ (R_-(\lambda)f)|_{\partial_-\mathcal{M}} = 0 \end{cases}, \quad \begin{cases} (-X + \lambda)R_+(\lambda)f = f \\ (R_+(\lambda)f)|_{\partial_+\mathcal{M}} = 0 \end{cases}$$

Extension to the complex plane

Theorem (Dyatlov-G '16)

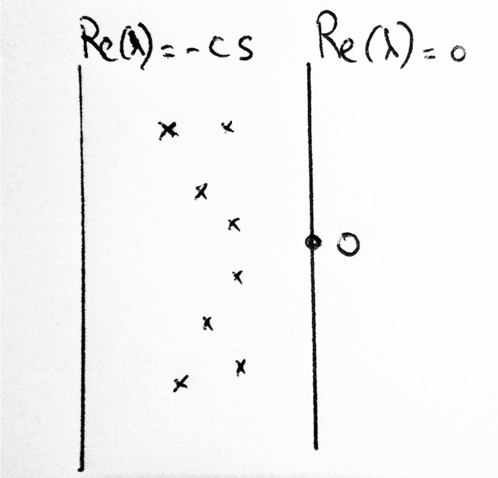
If the trapped set K is hyperbolic, there exists $c > 0$ so that for each $s > 0$ the operator $R_{\pm}(\lambda)$ extends to $\mathrm{Re}(\lambda) > -cs$ meromorphically in λ with finite rank poles, it maps

$$H_0^s(\mathcal{M}) \to H^{-s}(\mathcal{M}).$$

We obtain a description of wave-front set of the integral kernel $R_{\pm}(\lambda; x, x')$ in terms of stable/unstable bundles and $\Gamma_{\pm}$.

In the Anosov case:

* meromorphic extension by Butterley-Liverani '07, Faure-Sjöstrand '11
* the wave-front set analysis done by Dyatlov-Zworski '16.


Re(λ) = -cs
Re(λ) = 0

Tools used for this result

- Microlocal calculus - analysis in phase space
- Escape functions/Lyapunov functions (cf. Faure-Sjöstrand)
- use of anisotropic Sobolev spaces (cf. Kitaev, Blank, Keller, Liverani, Gouëzel, Baladi, Tsujii, Faure, Roy, Sjöstrand, etc): positive regularity in stable direction, negative in unstable.
- Set up of a Fredholm theory for the operator $X \pm \lambda$
- Propagation estimates: Hörmander propagation + propagation at radial sets (cf. Melrose, Vasy, Dyatlov-Zworski)

Applications to previous theorems (through linearized operator)

Lens rigidity problem:

1) let $\mathcal{M} = SM$, $\lambda = 0$: we deduce (microlocal) regularity of solutions of $Xu = f$ with $f \in C^\infty(\mathcal{M})$ satisfying $u|_{\partial_\pm \mathcal{M}} = 0$. This is the key for description of $\ker I_2$ in trapped case.

2) Deduce that the operator $I_2^* I_2 = \pi_*(R_+(0) - R_-(0))\pi^*$ is an elliptic pseudo-differential operator of order -1 on $(\ker I_2)^\perp$:

$I_2^* I_2 f \simeq \Delta_g^{-1/2} f + \mathrm{LOT}(f) \Longrightarrow$ stability estimates for I_2: $\|I_2 f\|_{L^2} \geq C\|f\|_{H^{-1/2}(M)}$

($\pi^* : C^\infty(M; S^2 T^* M) \to C^\infty(SM)$ natural operator, π_* its adjoint)

Marked length spectrum rigidity:

Use the operator $\Pi := R_+(0) - R_-(0)$ in Anosov case.

It is related to I_2 through Livsic theorem: let $f \in C^\alpha(SM)$,

$$\forall \gamma \in \mathcal{G}, \int_\gamma f = 0 \Longrightarrow \exists u \in C^\alpha(SM),\ f = Xu.$$

It satisfies stability estimates: for $f \in C^\alpha(M; S^2T^*M)$ with $f \perp \ker I_2$

$$C\|I_2 f\|^{1/2}_{\ell^\infty(\mathcal{G})}\|f\|^{1/2}_{C^\alpha(M)} \geq \|\pi_*\Pi\pi^* f\|_{L^2(M)} \geq \frac{1}{C}\|f\|_{H^{-1}(M)}.$$

Then since $0 = L(g)/L(g_0) - 1 = I_2(g - g_0) + O(\|g - g_0\|^2_{C^3})$, we get with $f := g - g_0$

$$\|f\|_{H^{-1}} \leq C\|I_2(g)\|^{1/2}_{\ell^\infty}\|f\|^{1/2}_{C^\alpha} \leq C\|f\|^{3/2}_{C^3}.$$

Merci!