From dynamics to topology via spectra

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Disclaimer

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Mathematical story where we meet the three subjects, a few friends and heroes who helped me to get an idea.

What is $1 + 1 + \cdots + 1 = ?$

9 years ago, I heard Sylvie Paycha start a talk with :

$$1 + 1 + \cdots + 1 + \cdots = ? \tag{1}$$

How and why should we study such problems ?

Motivations : number theory and physics.

Casimir effect in quantum physics. *Vacuum energy* in QFT, infinite sums :

$$\langle 0|H|0\rangle = C \sum_{\lambda\in \mathrm{Spec}} \lambda \tag{2}$$

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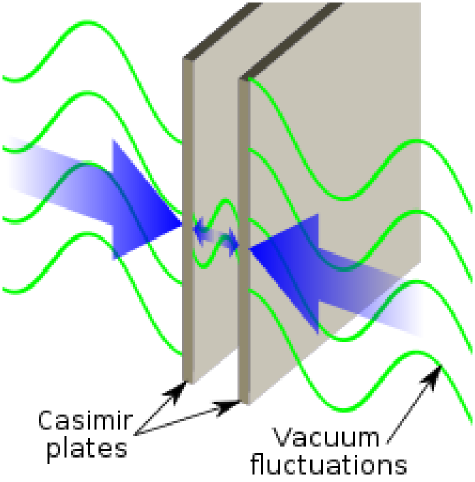
H Hamiltonian of the theory, an **operator dictating the evolution of the quantum system**, $|0>$ denotes the *vacuum state of the theory*, a vector in the state space describing the system and the sum runs over the spectrum of some operator.



[Casimir]

How to make sense of divergent sums ?

$\sum_{n=1}^{\infty} 1$ or $\sum_{\lambda \in \text{Spec}} \lambda$, counting an object, but divergent !

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$\sum_{n=1}^{\infty} 1$ or $\sum_{\lambda \in \mathrm{Spec}} \lambda$, counting an object, but divergent ! Heuristics of zeta regularization :

1. Some set $\mathcal{E}$ with *norm* $\|.\|$ measures **size** of objects,
2. Counting $N_T = |\{a \in \mathcal{E} \text{ s.t. } \|a\| \leqslant T\}|$,
3. Complex function associated to counting function, for example

$$\zeta(s) = \sum_{a \in \mathcal{E}} \|a\|^{-s} \text{ or } \eta(s) = \sum_{a \in \mathcal{E}} e^{-s\|a\|}$$

4. Regularized value = special value of complex function.

Zeta regularization of an infinite sum.

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (3)$$

Theorem

Riemann : *ζ admits a meromorphic continuation to $\mathbb{C}$.*

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Theorem

Riemann : *ζ admits a meromorphic continuation to $\mathbb{C}$.*
Euler : *ζ has special value at even integers*

$$\boxed{\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k},} \tag{4}$$

B_{2k} Bernouilli numbers, in particular $\zeta(0) = -\frac{1}{2}$.

A dictionary.

objects	Prime numbers	Eigenvalues Δ
counting function	$N_T = \|\{p \leqslant T\}\|$	$N_T = \|\{\lambda \leqslant T\}\|$
asymptotics	$N_T \sim \frac{T}{\log(T)}$ Hadamard	$N_T \sim CT^{\frac{d}{2}}$ Weyl
complex function	$\zeta(s) = \sum_1^\infty n^{-s}$ $= \prod_p (1-p^{-s})$ Riemann zeta	$\zeta_\Delta(s) = \sum \lambda^{-s}$ spectral zeta
conv domain	$Re(s) > 1$	$Re(s) > \frac{d}{2}$
analytic cont.	Riemann	Seeley
zeroes	?	?
special value	$\zeta(0) = 1 + \cdots + 1 + \cdots = -\frac{1}{2}$	$\zeta_\Delta(-\frac{1}{2})$ = Casimir energy

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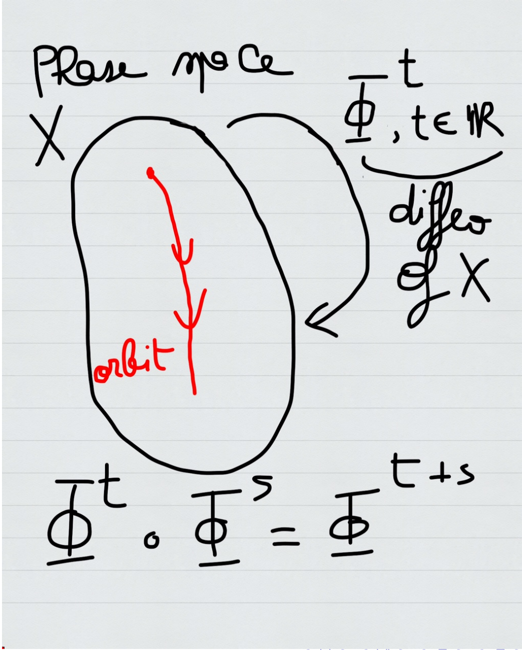
Divergent count in dynamics ?



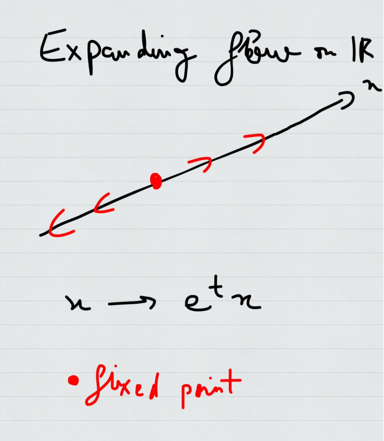
[Dynamics flow]

Hyperbolic dynamics, example 0.

Example

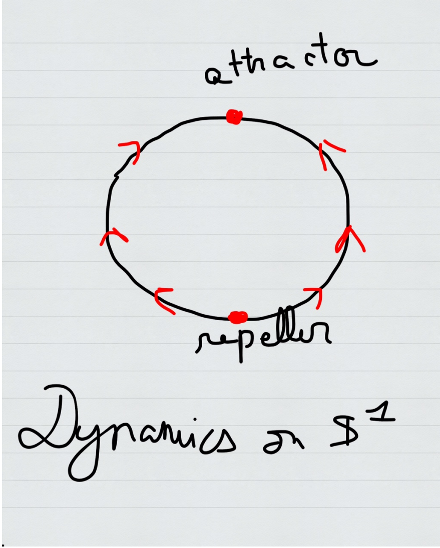
On $\mathbb{R}$, $x \mapsto e^t x$. **Expanding**

One repeller.



Hyperbolic dynamics, example 0.

Imagine attractor is at infinity, compactified in $\mathbb{S}^1$.



[Compactified version] .

Geodesic flow, example 1.

Geodesic flow on phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ position x, velocity v. Motion of free particle at x when $t = 0$ and initial velocity v. On $T\mathbb{R}^d$, $t \mapsto (x + tv, v)$.

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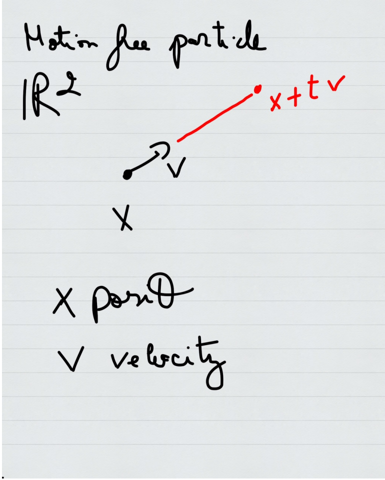
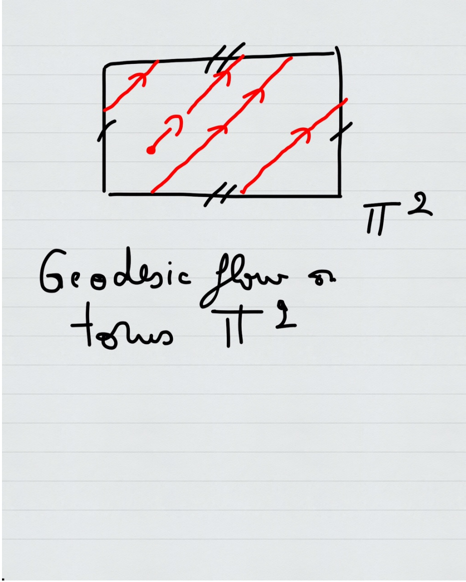


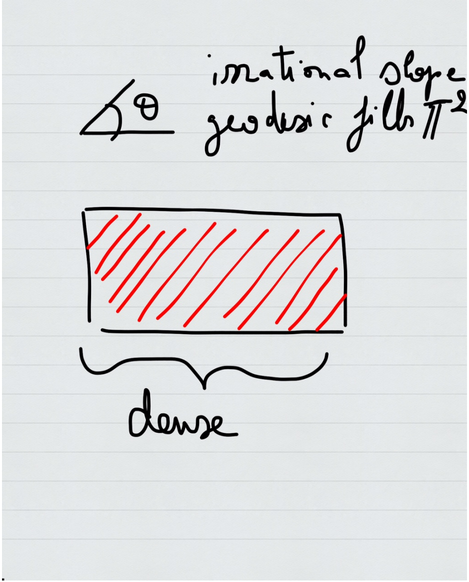
[Geodesic flow on plane]

What if we compactify $\mathbb{R}^d$?



[Geodesic flow torus] .

Ergodic : most orbits distribute equally on phase space, one says **equidistribute**.



[Equidistribution torus] .

Hyperbolic dynamics, example 2.

On a surface X with Riemannian metric g of negative curvature, what is a geodesic arc ?

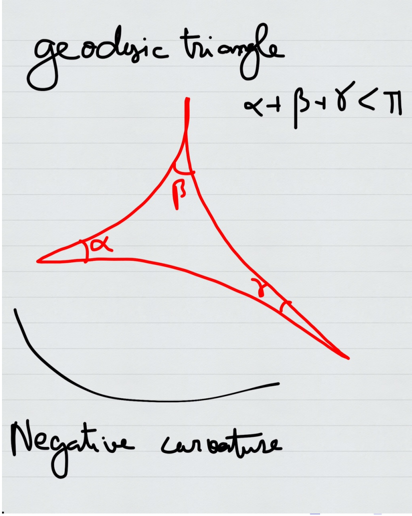
[Geodesic arc] .

Hyperbolic dynamics, example 3.

Surface X, with metric g of negative curvature.

Hyperbolic dynamics, example 3.

Surface X, with metric g of negative curvature. What does it mean negative curvature ?



[Geodesic triangle] .

Geodesic flow on SX hyperbolic and ergodic.

Theorem (Anosov)

The geodesic flow on SX is **ergodic***, in fact it is a consequence of being Anosov (no joke) !*

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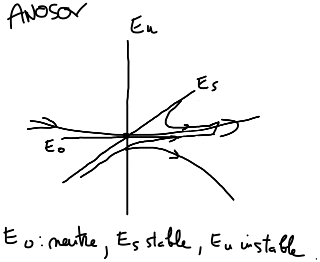
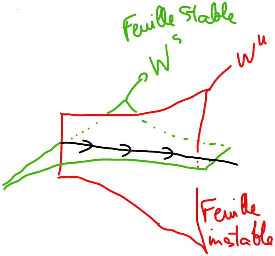
What is Anosov ?

Anosov



[Anosov decomposition of TSX]



[Stable and unstable foliations]

Omri Sarig's thought experiment.

A thought experiment. Drop a bit of ink into a glass of water, then stir it with a spoon.

- Can you predict where individual ink particles will end after 1 min ?

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- NO : the motion of ink particles is chaotic.
- Can you predict the density of the ink particles after 1 min ?
- YES : it will be nearly constant, equal to $\frac{|\text{ mass of ink }|}{|\text{volume of water+ink}|}$.

Transfer operators : motivation.

Gibbs's insight : For chaotic systems, it is often easier to predict the behavior of densities of large collections of initial conditions, then to predict the behavior of individual initial conditions.

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The transfer operator : The action of a dynamical system on **mass densities, extended objects**.

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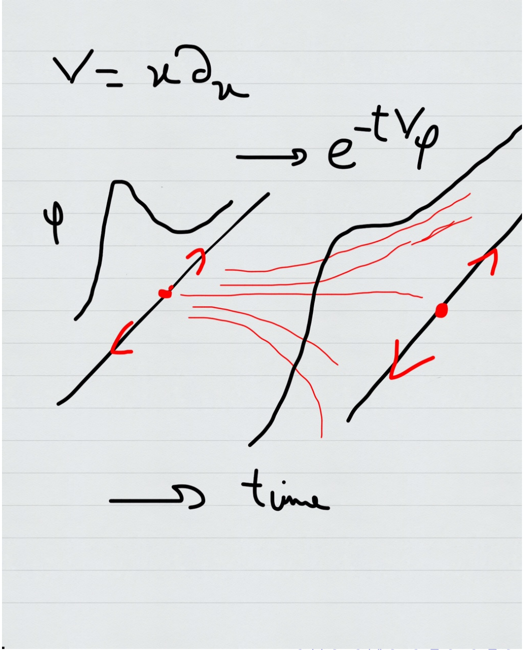
Mathematical setup ?



[transfer operator] .

Functional formalism of classical mechanics.

	Classical	functional formalism	Quantum
configuration space	(M,μ) mfd, measure	$C^\infty(M)$	$\mathcal{H}=L^2(M,d\mu)$ Hilbert space
dynamics generator	$\frac{d\Phi^t}{dt}=V\circ\Phi^t$ V vector field	$i\mathcal{L}_V$ Lie derivative	Δ Laplacian
Group	Φ^t flow	e^{-tV} transfer operator	$e^{it\Delta}$ propagator
information	$\Phi^t(x)$ trajectory	$\left\langle\psi_1,e^{-tV}\psi_2\right\rangle$ dynamical correlator	$\left\langle\psi_1,e^{it\Delta}\psi_2\right\rangle$ matrix coeff

Viewing spectras of matrices.

H a matrix, ψ_1, ψ_2 some test vectors, consider matrix element

$$\left\langle \psi_1, e^{-tH}\psi_2 \right\rangle = \sum_{\lambda \in \mathrm{Spec}(H)} e^{-t\lambda} P(\lambda, \psi_1, \psi_2) \quad (5)$$

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To capture large time t behaviour, Laplace transform

$$\boxed{\mathcal{L}C_{\psi_1,\psi_2}(z) = \int_0^\infty \left(\left\langle \psi_1, e^{-tH}\psi_2 \right\rangle\right) e^{-tz} dt = \left\langle \psi_1, (H+z)^{-1} \psi_2 \right\rangle .} \tag{6}$$

Poles of $\mathcal{L}C_{\psi_1,\psi_2} = -$ spectrum of H.

Pollicott–Ruelle resonances

On compact manifold M for $H = V$ Anosov vector field, ψ_1, ψ_2 test functions, poles of

$$\boxed{\mathcal{L}C_{\psi_1,\psi_2}(z) = \int_0^\infty \left(\int_M \psi_1 e^{-tV}\psi_2 d\mu\right) e^{-tz} dt = \left\langle \psi_1, (H+z)^{-1}\psi_2 \right\rangle,} \qquad (7)$$

are called **Pollicott–Ruelle** resonances.

Why study dynamical spectras ?

Assume $\sigma(V) = \{\underbrace{0 < \lambda_1} \leqslant \lambda_2 \leqslant \ldots\}$ and $\ker(V) =$ constant functions.

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$\sigma\left(e^{-tV}\right) = \{1 > e^{-t\lambda_1} \geqslant \dots\}$.

- Density $\psi_1 \in L^2(M, d\mu)$ of particles at entrance
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- Density $\psi_1 \in L^2(M, d\mu)$ of particles at entrance
- Domain Ω, 1_Ω indicator function
- *How many particles in Ω at time t ?*
- We find :

$$\begin{aligned}
\int_\Omega \left(\psi_1 \circ \Phi^{-t}\right) d\mu &= \langle 1_\Omega, e^{-tV}\psi_1\rangle \\
&= \underbrace{\langle 1_\Omega, \frac{1}{Vol(M)^{\frac{1}{2}}}\rangle\langle \frac{1}{Vol(M)^{\frac{1}{2}}}, \psi_1\rangle}_{\text{projects on } \ker(V)} + O\left(e^{-\lambda_1 t}\right) \\
&= \frac{Vol(\Omega)}{Vol(M)} \int_M \psi_1 d\mu + O\left(e^{-\lambda_1 t}\right)
\end{aligned}$$

Exponential convergence to Nb particles $\times \frac{Vol(\Omega)}{Vol(M)}$.

Dynamical features of spectras.

Spectral features $\implies$ density ψ_1 will *equidistribute in M by mixing uniformly* i.e. ergodic and exponentially mixing dynamics.

Klingen–Siegel Theorems.

Theorem (Hecke, Klingen–Siegel, Shintani refined by Deligne-Ribet, Cassou-Noguès)

Let $\mathfrak{f}$ and $\mathfrak{b}$ be two relatively prime ideals in the ring of integers $\mathcal{O}_F$. The partial zeta function attached to the ray class $\mathfrak{b}$ mod $\mathfrak{f}$ is defined by

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{\mathfrak{a}=\mathfrak{b} \mod (\mathfrak{f})} \frac{1}{\mathbf{N}(a)^s}, Re(s) > 1 \tag{8}$$

where $\mathfrak{a}$ runs over all integral ideals in $\mathcal{O}_F$ such that the fractional ideal $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal generated by a totally positive number in the coset $1 + \mathfrak{f}\mathfrak{b}^{-1}$. Then

$$\boxed{\zeta(\mathfrak{b}, \mathfrak{f}, 0) \in \mathbb{Q}.} \tag{9}$$

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Hard to understand but

Arithmetic results with knot theoretic interpretation.

Theorem (Bergeron-Charollois-Garcia-Venkatesh)

The special value $\zeta(\mathfrak{b},\mathfrak{f},0)$ can be interpreted as a linking number of periodic orbits in some 3–manifolds obtained by suspension of a linear automorphism of a torus.

A result of similar flavour,

Theorem (Ghys, Duke-Imamoglu-Toth)

On the unit tangent bundle of the modular surface $SL_2(\mathbb{Z})\setminus SL_2(\mathbb{R})$, linking of a closed geodesic and the trefoil knot can be identified with the value of the Rademacher function on the closed geodesic.

A result inspiring us.

Theorem (Dyatlov–Zworski)

X surface with negative curvature. Then

$$\zeta(s) = \prod_{\gamma}(1 - e^{-s\ell(\gamma)}) \tag{10}$$

product over prime periodic orbits γ of the geodesic flow, has meromorphic continuation on $\mathbb{C}$ (also Giuletti-Liverani-Pollicott).

$$\zeta(s) = s^{-\chi(X)}(C + \mathcal{O}(s)) \tag{11}$$

hence lenght of periodic geodesics gives genus of X.

[Gabriel Rivière]

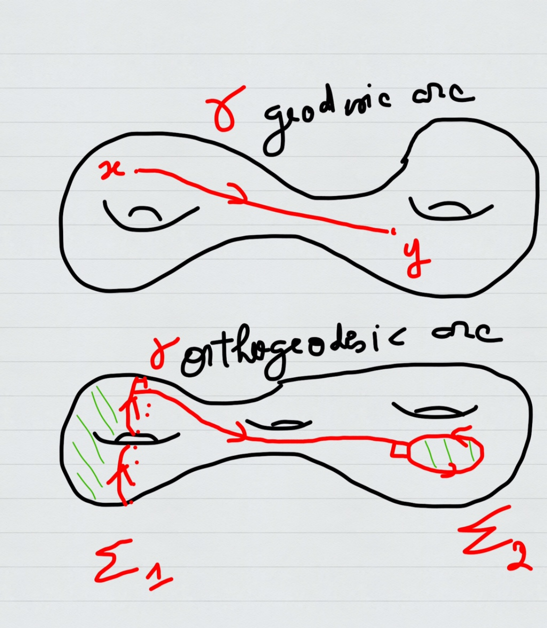
Poincaré series.

Surface X with negative curvature, (x, y) pair of points on X, consider

$$\eta(s) = \sum_{\gamma} e^{-s\ell(\gamma)} \tag{12}$$

where the sum runs over geodesic arcs $x \to y$.
More generally,

[Geodesic and orthogeodesic arcs]



Poincaré series.

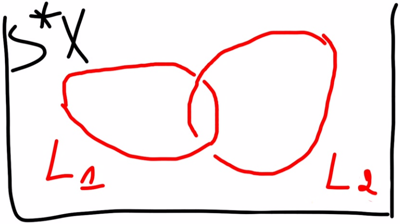
Surface X with negative curvature, (Σ_1, Σ_2) pair of closed geodesic curves on X, consider

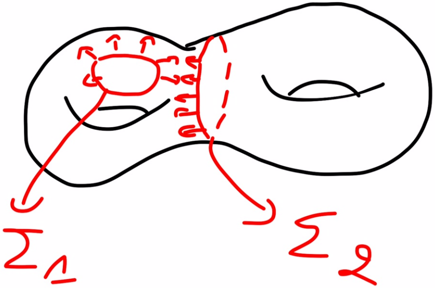
$$\eta(s) = \sum_{\gamma} e^{-s\ell(\gamma)}, \quad Re(s) > h_{top} \tag{13}$$

where the sum runs over **orthogeodesic arcs** $\Sigma_1 \to \Sigma_2$. η appears in Margulis, Pollicott, Sharp, Paternain, Mañé ...

Theorem (D-Rivière)

- *η has analytic continuation to the complex plane.*
- *Poles of $\eta \subset$ Pollicott-Ruelle resonances of the geodesic flow on SX*
- *When Σ_1, Σ_2 are homologically trivial, no poles at $s = 0$.*
- $\eta(0) = 1 + \cdots + 1 + \cdots \in \mathbb{Q}$ *explicit rational value obtained as* **linking number of two Legendrian knots**.





[Linking of Legendrian knots] .

Main idea of proof.

L_1, L_2 two Legendrian curves lifting Σ_1, Σ_2 to cotangent. Then $[L_1], [L_2]$ two integration currents :

$$\begin{aligned} \sum_\gamma e^{-s\ell(\gamma)} &= \int_0^\infty \left\langle [L_1], i_V e^{-tV}[L_2] \right\rangle e^{-ts} dt \\ &= \left\langle [L_1], i_V (V+s)^{-1} [L_2] \right\rangle . \end{aligned}$$

When $s = 0$, $\langle [L_1], i_V V^{-1}[L_2] \rangle$ is a correlation function where $i_V V^{-1}$ similar to Chern–Simons propagator, gives linking.

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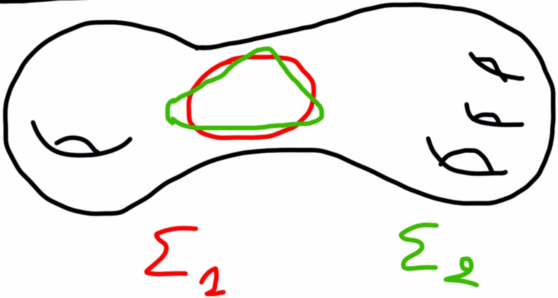
complex function	$\zeta(s) = \sum_1^\infty n^{-s}$ $= \prod_p (1-p^{-s})$ Riemann zeta	$\eta(s) = \sum_\gamma e^{-s\ell(\gamma)}$ Poincaré series
conv domain	$Re(s) > 1$	$Re(s) > h_{top}$
analytic cont.	Riemann	Selberg in curvature -1 D-Rivière
zeroes	?	Pollicott-Ruelle
$s = 0$	$\zeta(0) = 1 + \cdots + 1 + \cdots = -\frac{1}{2}$	$\eta(0) = 1 + \cdots + 1 + \cdots = \frac{1}{\chi(X)}$

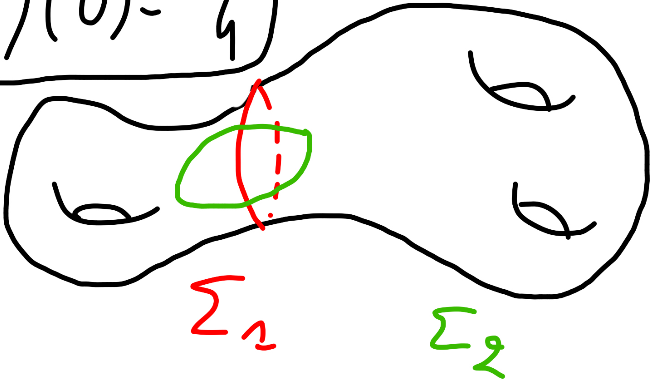
Thanks for your attention !

$$\eta(D) = -\frac{1}{6} + 2$$



$$\chi(X) = -6$$

$$\eta(0) = \frac{3}{4}$$



$$\chi(X) = -4$$